

DISCONTINUOUS DYNAMICAL SYSTEMS AND FRACTIONAL-ORDERS DIFFERENCE EQUATIONS

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ABSTRACT. In this work we are concerned with the definition of discrete dynamical systems of the fractional-orders difference equations. Then we use the discontinuous dynamical systems approach to study some dynamical behavior of these new dynamical systems.

1. INTRODUCTION

Consider the discrete dynamical system of the difference equation

$$x_n = f(x_{n-1/2}), n = 1, 2, 3, \dots \quad (1)$$

This difference equation can be written as

$$x_n = f(f(x_{n-1})) = f \circ f(x_{n-1}) = f^{(2)}(x_{n-1}) = g(x_{n-1}), n = 1, 2, 3, \dots \quad (2)$$

So, the discrete dynamical system of the difference equation (1) is equivalent to the one of the difference equation (2).

As an example, let $f(x) = \rho x(1-x)$ and consider the discrete dynamical system of the Logistic map

$$x_n = \rho x_{n-1/2}(1-x_{n-1/2}), n = 1, 2, 3, \dots \text{ and } x_0 = 0.3 \text{ (say)}. \quad (3)$$

Then the solution of (3) is given by

$$x_n = f^{(2n)}(x_0), n = 1, 2, 3, \dots$$

and the chaos and bifurcations of the solution is given in fig1.

Also the difference equation

$$x_n = f(x_{n-1/3}), n = 1, 2, 3, \dots \quad (4)$$

can be written as

$$x_n = f(f(f(x_{n-1}))) = f \circ f \circ f(x_{n-1}) = f^{(3)}(x_{n-1}) = h(x_{n-1}), n = 1, 2, 3, \dots$$

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and the discontinuous dynamical system of the difference equation (4) is equivalent to the one of the difference equation

$$x_n = h(x_{n-1}), \quad n = 1, 2, 3, \dots \quad (5)$$

For another example, let $f(x) = \rho x (1 - x)$ and consider the discrete dynamical system of the Logistic map

$$x_n = \rho x_{n-1/3} (1 - x_{n-1/3}), \quad n = 1, 2, 3, \dots \text{ and } x_0 = 0.3 \text{ (say)}. \quad (6)$$

Then the solution of (6) is given by

$$x_n = f^{(3n)}(x_0), \quad n = 1, 2, 3, \dots$$

and the chaos and bifurcations of the solution is given in fig 2.

Now, for the discontinuous dynamical system of the fractional-orders difference equation

$$x_n = f(x_{n-1/2}, x_{n-1}), \quad n = 1, 2, 3, \dots \quad (7)$$

and

$$x_n = f(x_{n-1/3}, x_{n-1}), \quad n = 1, 2, 3, \dots \quad (8)$$

we have another situation.

Our aim here is to apply the discontinuous dynamical systems approach, ([2]-[6]) to study some of the dynamical behavior, chaos and bifurcations, of the two discrete dynamical systems of the fractional-orders difference equations of the Logistic map

$$x_n = \rho x_{n-1/2} (1 - x_{n-1}), \quad n = 1, 2, 3, \dots \quad (9)$$

$$x_n = \rho x_{n-1/3} (1 - x_{n-1}), \quad n = 1, 2, 3, \dots \quad (10)$$

Firstly, we apply the discontinuous dynamical systems approach for the two discrete dynamical systems (3) and (6) and obtain the same previous results (see graph 3 and graph 4). Then we apply this approach to the discrete dynamical systems of the fractional-orders Logistic map (9) and (10).

2. DISCONTINUOUS DYNAMICAL SYSTEMS

Consider the problem of retarded functional equation

$$x(t) = f(x(t-r)), \quad t \in (0, T], \text{ and } x(0) = x_0 \quad (11)$$

Let $t \in (0, r]$, then $t - r \in (-r, 0]$ and the solution of (11) is given by

$$x(t) = x_r(t) = f(t, x_0), \quad t \in (0, r].$$

For $t \in (r, 2r]$, we find that $t - r \in (0, r]$ and the solution of (11) is given by

$$x(t) = x_{2r}(t) = f(x_r(t)) = f^2(x_0), \quad t \in (r, 2r].$$

Repeating the process we can deduce that the solution of the problem (11) is given by

$$x(t) = x_{nr}(t) = f^n(x_0), \quad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr)$, $k = 1, 2, \dots, n$, but

$$\lim_{t \rightarrow kr^+} x_{(k+1)r}(t) = f^{k+1}(x_0) \neq x_{kr}(t),$$

which implies that the solution of the problem (11) is discontinuous (sectionally continuous) on $(0, T]$ and we have proved the following theorem

Theorem 1 The solution of the problem of retarded functional equation is discontinuous (sectionally continuous) even the function f is continuous.

Now, let $f : [0, T] \times R^n \rightarrow R^n$ and $r_1, r_2, \dots, r_n \in R_+$. Then we can give the following definition

Definition 1 The discontinuous dynamical system is the problem of retarded functional equation

$$x(t) = f(t, x(t-r_1), x(t-r_2), \dots, x(t-r_n)), \quad t \in (0, T], \quad (12)$$

$$x(t) = x_0, \quad t \leq 0 \quad (13)$$

Example 1 The discontinuous dynamical system corresponding to the Logistic equation is

$$x(t) = \rho x(t-r) (1 - x(t-r)), \quad t \in (0, T],$$

$$x(t) = x_0, \quad t \leq 0.$$

The corresponding discrete dynamical system is

$$x_n = \rho x_{n-1} (1 - x_{n-1}), \quad n = 1, 2, \dots, N,$$

$$x_{-1} = x_0.$$

Example 2 The discontinuous dynamical system corresponding to the Logistic equation with two different delays is

$$x(t) = \rho x(t-r_1) (1 - x(t-r_2)), \quad t \in (0, T],$$

$$x(t) = x_0, \quad t \leq 0.$$

A corresponding discrete dynamical system is

$$x_n = \rho x_{n-1} (1 - x_{n-2}), \quad n = 1, 2, \dots, N,$$

$$x_{-1} = x_0.$$

3. FRACTIONAL-ORDERS SYSTEMS

Let $r = \frac{1}{2}$ and consider the discontinuous dynamical system

$$x(t) = \rho x(t-r) (1 - x(t-r)), \quad t \in (0, T] \text{ and } x(t) = x_0, \quad t \leq 0. \quad (14)$$

Let $t \in (0, r]$, then $(t-r) \in (-r, 0]$ and the solution of (14) is given by

$$x_r(t) = \rho x_0 (1 - x_0), \text{ and } x_r(r) = \rho x_0 (1 - x_0) = x_r.$$

Let $t \in (r, 2r]$, then $(t-r) \in (0, r]$ and the solution of (14) is given by

$$x_{2r}(t) = \rho x_r (1 - x_r), \text{ and } x_{2r}(2r) = \rho x_r (1 - x_r) = x_{2r},$$

then

$$x_1 = \rho x_r (1 - x_r) = x_{2r}. \quad (15)$$

Let $t \in (2r, 3r]$, then $(t-r) \in (r, 2r]$ and the solution of (14) is given by

$$x_{3r}(t) = \rho x_{2r} (1 - x_{2r}), \text{ and } x_{3r}(3r) = \rho x_{2r} (1 - x_{2r}) = x_{3r}.$$

Let $t \in (3r, 4r]$, then $(t-r) \in (2r, 3r]$ and the solution of (14) is given by

$$x_{4r}(t) = \rho x_{3r} (1 - x_{3r}), \text{ and } x_{4r}(4r) = \rho x_{3r} (1 - x_{3r}) = x_{3r},$$

then

$$x_2 = \rho x_{3r} (1 - x_{3r}). \quad (16)$$

Repeating the process we can deduce that the solution of the problem (14) is given by

$$x_{mr}(t) = \rho x_{(m-1)r} (1-x_{(m-1)r}), \text{ and } x_{mr}(mr) = \rho x_{(m-1)r} (1-x_{(m-1)r}) = x_{mr}$$

Now for the discrete dynamical system (3), the solution is given by

$$x_{m/2} = \rho x_{(m-1)/2} (1 - x_{(m-1)/2}), \quad m = 2, 4, 6, \dots \quad (17)$$

and

$$x_n = \rho x_{(m-1)/2} (1 - x_{(m-1)/2}), \quad n = 1, 2, 3 \dots \quad (18)$$

The chaos and bifurcation of this system is given by Figure 3, which is the same as Figure 1.

By the same way we can obtain the solution of the discrete dynamical system (6) by

$$x_{m/3} = \rho x_{(m-1)/3} (1 - x_{(m-1)/3}), \quad m = 3, 6, 9, \dots \quad (19)$$

and

$$x_n = \rho x_{(m-1)/3} (1 - x_{(m-1)/3}), \quad n = 1, 2, 3 \dots \quad (20)$$

The chaos and bifurcation of this system is given by Figure 4, which is the same as Figure 3.

3.1. General case. Now for the problems (9) and (10) we have the following.

Let $r = \frac{1}{2}$ and consider the discontinuous dynamical system

$$x(t) = \rho x(t-r) (1 - x(t-1)), \quad t \in (0, T] \text{ and } x(t) = x_o, \quad t \leq 0. \quad (21)$$

Let $t \in (0, r]$, then $(t-r) \in (-r, 0]$ and the solution of (21) is given by

$$x_r(t) = \rho x_o (1 - x_o), \text{ and } x_r(r) = \rho x_o (1 - x_o) = x_r.$$

Let $t \in (r, 2r]$, then $(t-r) \in (0, r]$ and the solution of (21) is given by

$$x_{2r}(t) = \rho x_r (1 - x_o), \text{ and } x_{2r}(2r) = \rho x_r (1 - x_o) = x_{2r}$$

and

$$x_1 = \rho x_r (1 - x_o). \quad (22)$$

Let $t \in (2r, 3r]$, then $(t-r) \in (r, 2r]$ and the solution of (21) is given by

$$x_{3r}(t) = \rho x_{2r} (1 - x_r), \text{ and } x_{3r}(3r) = \rho x_{2r} (1 - x_r) = x_{3r}.$$

Let $t \in (3r, 4r]$, then $(t-r) \in (2r, 3r]$ and the solution of (21) is given by

$$x_{4r}(t) = \rho x_{3r} (1 - x_{2r}), \text{ and } x_{4r}(4r) = \rho x_{3r} (1 - x_{2r}) = x_{3r}.$$

and

$$x_2 = \rho x_{3r} (1 - x_{2r}). \quad (23)$$

Repeating the process we can deduce that the solution of the problem (21) is given by

$$x_{mr}(t) = \rho x_{(m-1)r} (1-x_{(m-2)r}), \text{ and } x_{mr}(mr) = \rho x_{(m-1)r} (1-x_{(m-2)r}) = x_{mr}.$$

Now for the discrete dynamical system (9), the solution is given by

$$x_{m/2} = \rho x_{(m-1)/2} (1 - x_{(m-2)/2}), \quad m = 2, 4, 6, \dots \quad (24)$$

and

$$x_n = \rho x_{(m-1)/2} (1 - x_{(m-2)/2}), \quad n = 1, 2, 3 \dots \quad (25)$$

The chaos and bifurcation of this system is given by Figure 5.

By the same way, the solution of the discrete dynamical system (10) is given by

$$x_{m/3} = \rho x_{(m-1)/3} (1 - x_{(m-3)/3}), \quad m = 3, 6, 9, \dots \quad (26)$$

and

$$x_n = \rho x_{(m-1)/3} (1 - x_{(m-3)/3}), \quad n = 1, 2, 3, \dots \quad (27)$$

The chaos and bifurcation of this system is given by Figure 6.

4. EQUILIBRIUM POINTS AND THEIR STABILITY

The equilibrium points of (9) are the solution of the equation

$$\rho x_{eq} (1 - x_{eq}) = x_{eq}$$

which are

$$\begin{aligned} (x_{eq})_1 &= 0, \\ (x_{eq})_2 &= 1 - \frac{1}{\rho}. \end{aligned}$$

To determine the stability of a fixed point, consider a small perturbation from the fixed point by letting

$$x_n = x_{eq} + \varepsilon_0 \lambda^n. \quad (28)$$

We can find the map for the deviation $\varepsilon_0 \lambda^n$ by substituting (28) into (9) to obtain

$$x_{eq} + \varepsilon_0 \lambda^n = \rho (x_{eq} + \varepsilon_0 \lambda^{n-\frac{1}{2}}) [1 - x_{eq} - \varepsilon_0 \lambda^{n-1}],$$

which implies that

$$1 = \rho [(1 - x_{eq}) \lambda^{-\frac{1}{2}} - x_{eq} \lambda^{-1}], \quad (29)$$

The equilibrium point of (9) is locally stable if all the roots λ of the equation (29) satisfy $|\lambda| < 1$ (see [8]).

Then the equilibrium point $x_{eq} = 0$ is locally stable if $\rho < 1$, while the second equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$ is locally stable if all the roots λ of the equation,

$$\lambda - \lambda^{\frac{1}{2}} + (\rho - 1) = 0, \quad (30)$$

satisfy $|\lambda| < 1$.

Then the equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$, $\rho > 1$ is locally stable if $1 < \rho < 2$.

By the same way we can obtain the Equilibrium Points and their stability for map (10).

5. BIFURCATION AND CHAOS

In this section, some numerical simulations results are presented to show that dynamical behavior for the two discrete dynamical systems (3) and (6) is the same dynamical behavior for the discontinuous dynamical systems (17) and (19) respectively. Also, we show the dynamical behavior for the two discrete dynamical systems (9) and (10) by using the discontinuous dynamical systems approach (24) and (26).

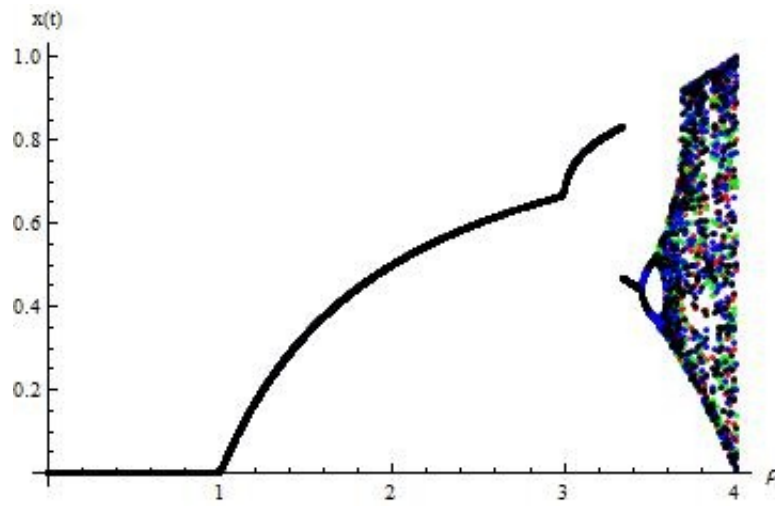


Figure 1: Bifurcation diagram of map (3) with respect to ρ .

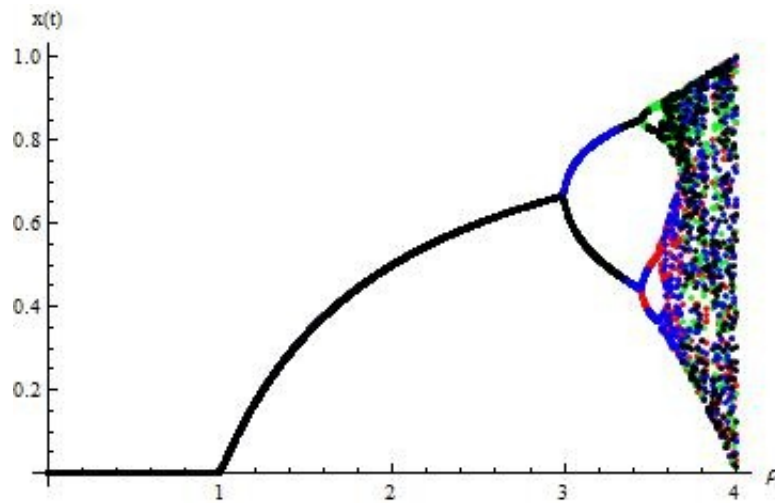


Figure 2: Bifurcation diagram of map (6) with respect to ρ .

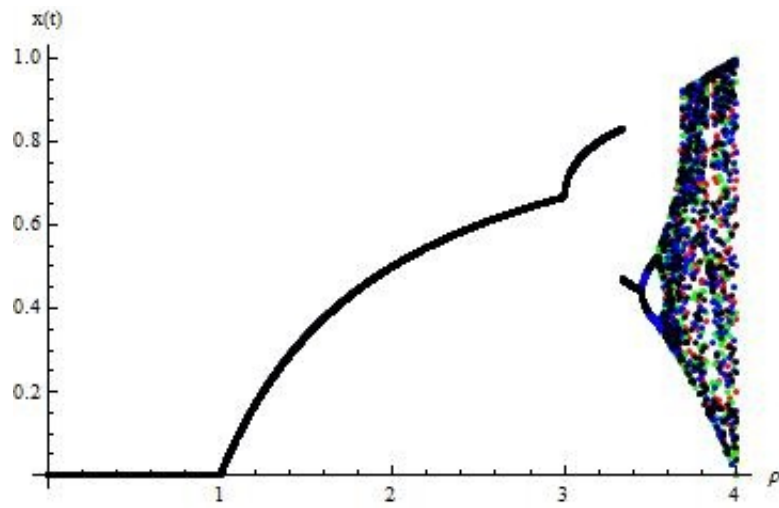


Figure 3: Bifurcation diagram of map (17) with respect to ρ .

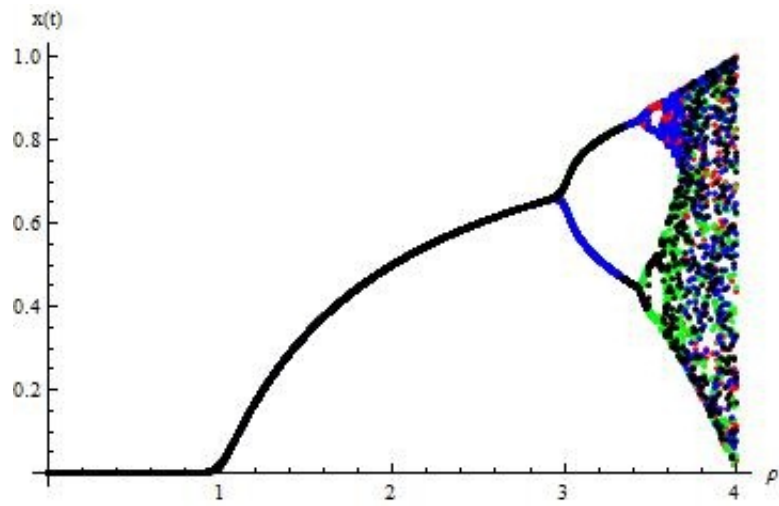


Figure 4: Bifurcation diagram of map (19) with respect to ρ .

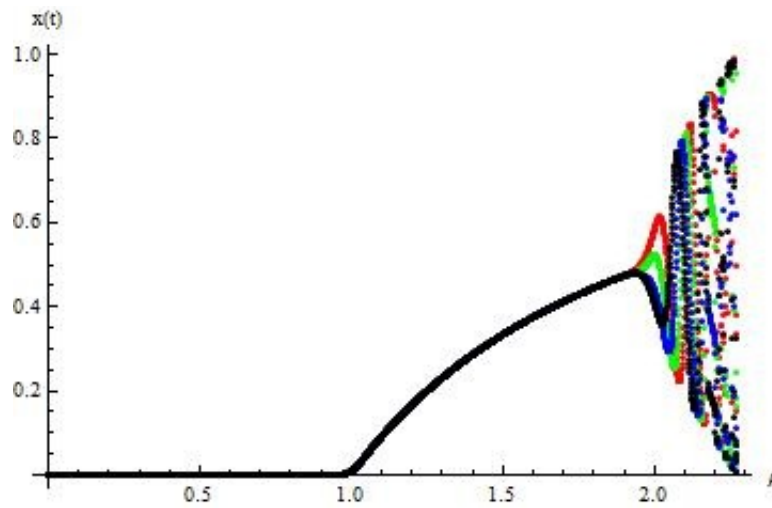


Figure 5: Bifurcation diagram of map (24) with respect to ρ .

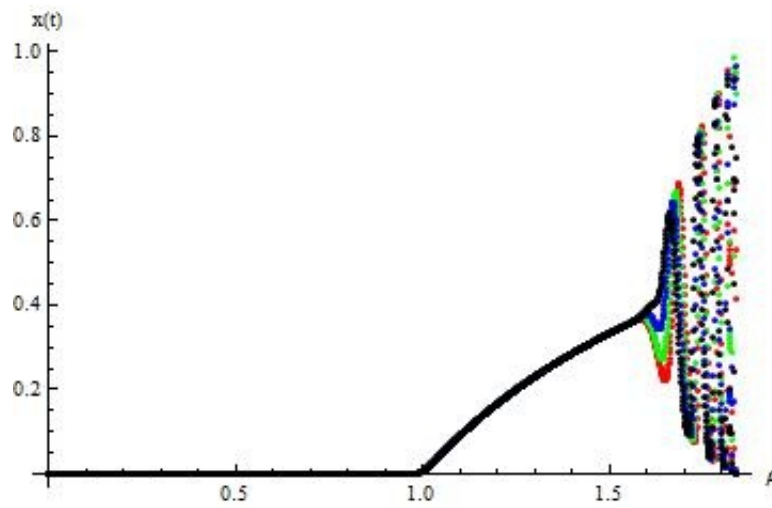


Figure 6: Bifurcation diagram of map (26) with respect to ρ .

6. CONCLUSIONS

Here we used the discontinuous dynamical systems approached to study the chaos and bifurcation of the discrete dynamical systems of fractional orders.

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